A penalty method applied to the fractional diffusion equation

Application of the penalty method

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Introduction

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- Almost always treated with low order finite difference method.
- Dealing with the boundary is not always trivial.
- We would like to be able to deal with obstacle and non-periodic domain.
- We want a high-order scheme without too much programming effort (prefferably a spectral method).

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- limit ourselves to real order fractional operators.

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Definition (Liouville integral)

Let $\nu > 0$. We define the Liouville fractional integral of order ν of $f(t)$ as :

$$ {}_{-\infty}J_t^{\nu}\left[f(t)\right] := \frac{1}{\Gamma(\nu)} \int_{-\infty}^{t} (t-\tau)^{\nu-1} f(\tau)\,\mathrm{d}\tau \tag{1} $$

If $\nu = 0$,

$$ {}_{-\infty}J_t^{0}\left[f(t)\right] := I[f(t)] = f(t) \tag{2} $$

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$$\begin{aligned} {}_{-\infty}D_t^{\mu}\left[f(t)\right] &= \frac{\mathrm{d}^m}{\mathrm{d}t^m}\left[{}_{-\infty}J_t^{\nu}\left[f(t)\right]\right] \\ &= \frac{1}{\Gamma(\nu)}\frac{\mathrm{d}^m}{\mathrm{d}t^m}\left[\int_{-\infty}^{t}(t-\tau)^{\nu-1}f(\tau)\,\mathrm{d}\tau\right] \end{aligned} \tag{3}$$

where $\nu = m - \mu$ and $m = \lfloor \mu \rfloor + 1$.
If $\nu = 0$,

$${}_{-\infty}D_t^{0}\left[f(t)\right] := I[f(t)] = f(t) \tag{4}$$

Some remarks

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Let ${}_{-\infty}D_t^{\alpha}\left[f(t)\right]$ be a Liouville fractional operator (derivative or integral). Then :

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The order is **important** ! The Liouville fractional derivative is a **left** inverse (in general).

Theorem

Let $\mathcal{F}$ be the Fourier transform operator. Then, for a sufficiently good function, we have :
for $0 < \nu < 1$,

$$(\mathcal{F}[{}_{-\infty}J_t^{\nu}[f(t)]])(\kappa) = \frac{(\mathcal{F}[f(t)])(\kappa)}{(-i\kappa)^{\nu}}$$

and, for $\mu > 0$,

$$(\mathcal{F}[{}_{-\infty}D_t^{\mu}[f(t)]])(\kappa) = (-i\kappa)^{\mu}(\mathcal{F}[f(t)])(\kappa)$$

Fourier transform of fractional derivative

Good news everyones ! We can use Fourier pseudo-spectral method in our problem.

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- Problem with regularity on the original boundary.
- Drastically lower the convergence order of the spectral method.
- Low order in function of the penalty parameter.
- Boundary layer gives low order of convergence in space.

Let's consider the problem :

$$\partial_t u = \partial_{xx} u \text{ in } \Omega_p$$

$$u = g \text{ on } \Gamma_p$$

where Ω_p is the physical domain and Γ_p is the boundary.

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In the original penalty method we extend our domain to a periodic one by adding a viscous term :

$$\partial_t u_\eta = \partial_{xx} u_\eta - \eta^{-1} \chi_s u_\eta \text{ in } \Omega$$

where η is a small penalty parameter and χ_s is the indicator function of the domain extension Ω_s

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where η is a small penalty parameter and χ_s is the indicator function of the domain extension Ω_s
We only have $u_\eta \to u$ with $\mathcal{O}(\sqrt{\eta})$ and in the boundary layer $\mathcal{O}(\Delta x)$.

We modify the penalty term by adding an extension function $\tilde{g}$ to reduce the boundary layer at the vignicity of Γ_p.

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We used Hermite interpolation to extend the boundary.

We shall consider the fractional heat equation with $1 < \alpha \leq 2$:

$$\partial_t u(x,t) = {}_{-\infty}D_x^{\alpha}\left[u(x,t)\right] + f(x,t) \text{ in } [0, 2\pi]$$

$$f(x,t) = -\cos(t)\sin(x + \pi\alpha/2) - \sin(t)\sin(x)$$

$$u(x,0) = \sin(x)$$

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For $5 \leq n \leq 10$ we will pose : $N = 2^n$, $h = \frac{2\pi}{N}$, $\Delta t = 0.1h^2$ and $\eta = 5\Delta t$. We will integrate to a final time $T = 1$.

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Also, we will do all tests for 0,1, 2 derivatives matched at the boundaries during the Hermite interpolation.

- We discretize u_t with a simple Euler scheme :

$$u_t^{n+1} = \frac{u^{n+1} - u^n}{\Delta t}$$

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- The Fourier spectral method is not of exponential order if we have discontinuities.
- To allow high order of convergence with our spectral method we smoothed out the indicator function.
- It is also important to have smooth initial conditions. We used Hermite interpolation to to achieve this.

Fractional heat equation $\nu = 1.5$

For this problem, we will consider an obstacle in the region $\Omega_s = [\pi - 0.7, \pi + 0.7]$

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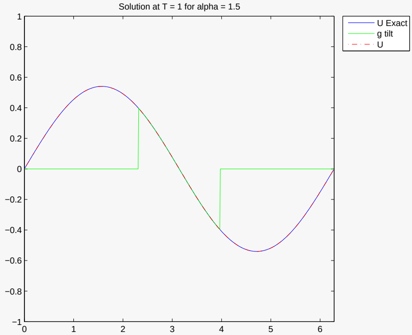

Figure : Solution for the fractional heat equation of order $\nu = 1.5$ with H_2

Fractional heat equation $\nu = 1.5$

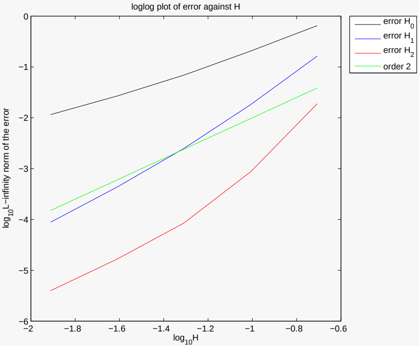

Figure : Loglog plot of the error versus H for the fractional heat equation of order $\nu = 1.5$

	C^0	C^1	C^2
Order	1.45	2.71	3.01

Fractional heat equation $\nu = 1.5$

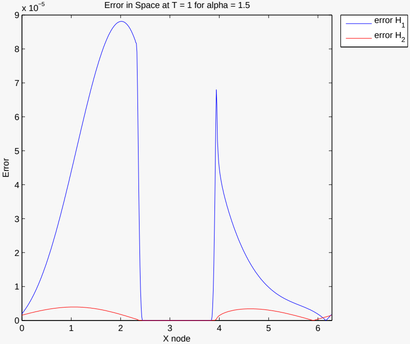

Figure : Error for the fractional heat equation of order $\nu = 1.5$

Non-periodic fractional heat equation $\nu = 1.5$

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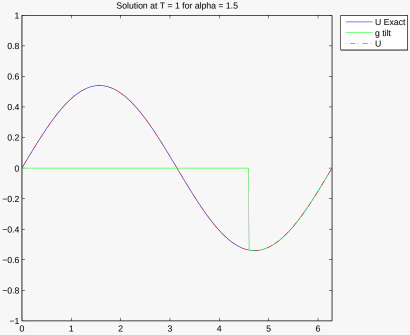

Figure : Solution for the non-periodic fractional heat equation of order $\nu = 1.5$ with H_2

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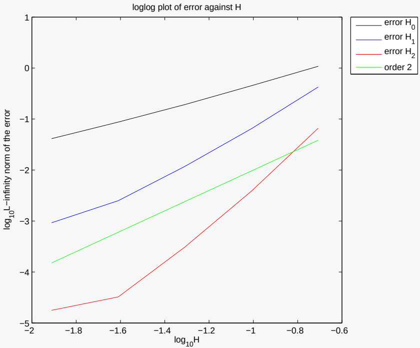

Figure : Loglog plot of the error versus H for the non-periodic fractional heat equation of order $\nu = 1.5$

	C^0	C^1	C^2
Order	1.18	2.24	3.06

Non-periodic fractional heat equation $\nu = 1.5$

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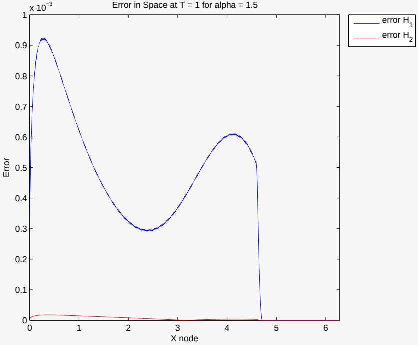

Figure : Error for the non-periodic fractional heat equation of order $\nu = 1.5$

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- It would be interesting to try other pseudo-spectral methods for various definitions of a fractional operator.

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